CALCULATION OF THE INTERACTION OF SUPERSONIC COUNTER FLOWS

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The maximum stable difference scheme of second order accuracy is described for axisymmetrical flows with shock waves and the results are given of a calculation of the interaction between an external supersonic flow and a counter supersonic jet issuing from a cylindrical channel.

1. Formulation of the Problem. In order to calculate gas flows with shock waves, difference methods of direct calculation have been proposed by different authors. defined by the order of accuracy, stability and degree of complexity in carrying out the calculation on a computer. It is found, by the practical application of direct numerical methods, that difference schemes of the first order frequently give a large error, and that schemes of the third and fourth order are too complex. Schemes of second order accuracy are compromises between the so-called classes of schemes and combine satisfactory accuracy with simplicity of execution.

Several versions of second-order difference schemes are known [1-4]. A scheme is described in this paper for axisymmetrical flows, which differs from the schemes in [1], [3], and [4] by the increased stability (the step can be increased by a factor of $\sqrt{2}$) and from the scheme in [2] by the possibility of using data about the field of gasdynamic parameters obtained on the previous as well as on the last half-step.

The proposed difference method is used to solve the problem of the interaction between a supersonic flow incident on a cylindrical body with a counter jet issuing from the body. The counter flow fulfills the role of an advancing curved profile, reducing the resistance to motion. As a result of this, in front of the end face of the streamlined cylinder a leading shock wave is formed, the contact surface separating the interacting flows, the counter shock wave and a vortex zone (Fig. 1). The determination of the dependence of the pressure (force loading) at the surface on the flow parameters and the geometry of the body is of practical interest.

The problem is solved by determining at the initial instant, the field of the parameters corresponding to the instantaneous injection of the cylinder into the supersonic flow, the process of transition to a steady-state distribution is then calculated, which is the solution of the streamline flow problem. When a steady-state streamline flow cycle is achieved, the boundary conditions at the end surface of the body are changed in such a way that the counter-supersonic flow would be simulated. After this, the solution of the problem is continued until a new steady-state cycle is obtained.

2. Two-Step Second-Order Difference Scheme. The mathematical model of nonsteady-axisymmetrical flow is a system of Euler equations with the appropriate initial and boundary conditions. The system of equations is described in divergent form by

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial x}F(f)+\frac{\partial}{\partial y}G(f)+\frac{\nu}{y}H(f)=0 \qquad (1)$$

using the vector notations

$$f=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ e\end{pmatrix},\ F=\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ (e+p)u\end{pmatrix},\ G=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ (e+p)v\end{pmatrix},\ H=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2\\ (e+p)v\end{pmatrix}.$$

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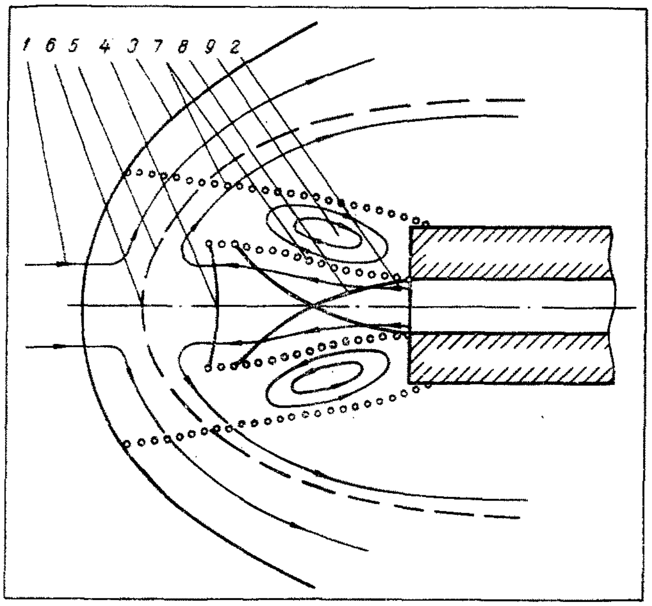

Fig. 1. Interaction diagram: 1) external flow; 2) counter-jet; 3) leading shock wave; 4) counter-shock wave; 5) contact surface; 6) stagnation point; 7) sonic lines; 8) shock waves in the jet; 9) subsonic vortex region.

The parameter ν defines the form of symmetry: $\nu = 1$ in the axisymmetrical case and $\nu = 0$ in the plane case.

In addition to the general system (1) for splitting the two-dimensional difference operator into one-dimensional operators, we introduce the supplementary one-dimensional systems:

$$\frac{\partial \vec{f}}{\partial t} + \frac{\partial}{\partial x} F(\vec{f}) = 0, \tag{2}$$

$$\frac{\partial \vec{f}}{\partial t} + \frac{\partial}{\partial y} G(\vec{f}) + \frac{\nu}{y} H(\vec{f}) = 0. \tag{3}$$

The difference approximation of the system is effected by the network

$$x_k = k \cdot \Delta x,\ k = 0,\ 1,\ 2,\ \ldots;\ y_l = l \cdot \Delta y,\ l = 0,\ 1,\ 2,\ \ldots;\ t^n = \sum_{i=0}^{n} \Delta t^i$$

with constant steps with respect to the space coordinates and with time-steps defined from the condition of stability [17]. For the approximation, nine points of the reference layer are included, one point of the denumerable layer and also four points of the intermediate layer with half-integral suffixes.

A four-point maximum-stable local one-dimensional scheme of the first order of accuracy is effected by a preliminary half-step. If we designate L_x as the difference operator, approximating by a half-step the vector equation (2) and we designate L_y as the operator approximating equation (3), then the operator of the first half-step can be represented as the symmetrical product of L_x and L_y:

$$L = \frac{1}{2}(L_x L_y + L_y L_x), \tag{4}$$

The averaging and difference operators, μ_x, μ_y and δ_x, δ_y respectively are introduced for compact notation of the working formulas

$$\mu_x f_{k,l} = \frac{1}{2}\left(f_{k-\frac{1}{2},\,l} + f_{k+\frac{1}{2},\,l}\right),\quad \mu_y f_{k,l} = \frac{1}{2}\left(f_{k,\,l-\frac{1}{2}} + f_{k,\,l+\frac{1}{2}}\right),$$

$$\delta_x f_{k,l} = f_{k+\frac{1}{2},\,l} - f_{k-\frac{1}{2},\,l},\quad \delta_y f_{k,l} = f_{k,\,l+\frac{1}{2}} - f_{k,\,l-\frac{1}{2}}.$$

Using these operators, the working formulas of the preliminary half-step, effecting relations (4), are represented in the form:

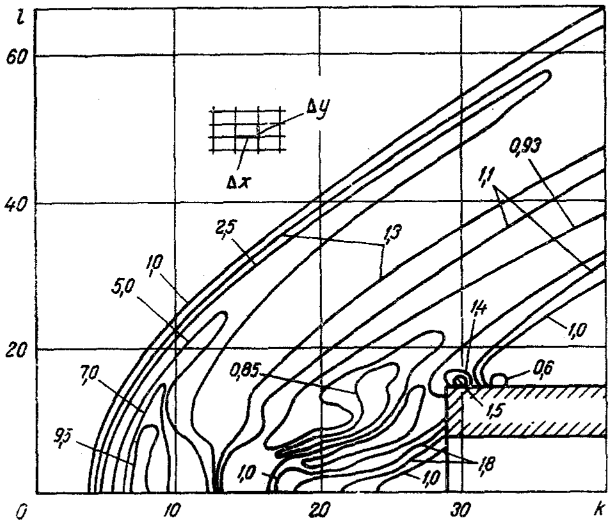

Fig. 2. Isobar field in the variant with $M_1 = 2.5$ and $M_2 = 3.25$.

$$\bar{f}_{k+\frac{1}{2},l} = \mu_x f^n_{k+\frac{1}{2},l} - \frac{\Delta t^n}{2\Delta x}\delta_x F^n_{k+\frac{1}{2},l},$$

$$\bar{\bar{f}}_{k,l+\frac{1}{2}} = \mu_y f^n_{k,l+\frac{1}{2}} - \frac{\Delta t^n}{2\Delta y}\delta_y G^n_{k,l+\frac{1}{2}} - \frac{\nu\Delta t^n}{2y_{l+\frac{1}{2}}}\mu_y H^n_{k,l+\frac{1}{2}},$$

$$\hat{f}_{k+\frac{1}{2},l+\frac{1}{2}} = \mu_x \bar{\bar{f}}_{k+\frac{1}{2},l+\frac{1}{2}} - \frac{\Delta t^n}{2\Delta x}\delta_x \bar{\bar{F}}_{k+\frac{1}{2},l+\frac{1}{2}},$$

$$\hat{\hat{f}}_{k+\frac{1}{2},l+\frac{1}{2}} = \mu_y \bar{f}_{k+\frac{1}{2},l+\frac{1}{2}} - \frac{\Delta t^n}{2\Delta y}\delta_y \bar{G}_{k+\frac{1}{2},l+\frac{1}{2}} - \frac{\nu\Delta t^n}{2y_{l+\frac{1}{2}}}\mu_y \bar{H}_{k+\frac{1}{2},l+\frac{1}{2}}, \tag{5}$$

$$f^{n+\frac{1}{2}}_{k+\frac{1}{2},l+\frac{1}{2}} = \frac{1}{2}\left(\hat{f}_{k+\frac{1}{2},l+\frac{1}{2}} + \hat{\hat{f}}_{k+\frac{1}{2},l+\frac{1}{2}}\right).$$

In the second and final half-step, a two-dimensional version of the "KREST" ("CROSS") scheme is used:

$$\bar{\bar{f}}^{n+1}_{k,l} = f^n_{k,l} - \frac{\Delta t^n}{\Delta x}\mu_y\delta_x F^{n+\frac{1}{2}}_{k,l} - \frac{\Delta t^n}{\Delta y}\mu_x\delta_y G^{n+\frac{1}{2}}_{k,l} - \frac{\nu\Delta t^n}{y_l}\mu_x\mu_y H^{n+\frac{1}{2}}_{k,l}. \tag{6}$$

In addition to the half-steps (5) and (6), in order to increase the quality of the calculation of the shock waves, the third stage — smoothing — is carried out. In this stage, the terms of the artificial viscosity of the third order of smallness are added to the solution obtained:

$$f^{n+1}_{k,l} = \bar{\bar{f}}^{n+1}_{k,l} + \delta_x Q^n_{k,l} + \delta_y R^n_{k,l},$$

$$Q^n_{k+\frac{1}{2},l} = \frac{\omega}{2}|\delta_x \varkappa^n_{k+\frac{1}{2},l}|\,\delta_x f^n_{k+\frac{1}{2},l},$$

$$R^n_{k,l+\frac{1}{2}} = \frac{\omega}{2}|\delta_y \lambda^n_{k,l+\frac{1}{2}}|\,\delta_y f^n_{k,l+\frac{1}{2}}, \tag{7}$$

$$\varkappa^n = \frac{\Delta t^n}{\Delta x}(|u| + \bar{a})^n,\quad \lambda^n = \frac{\Delta t^n}{\Delta y}(|v| + \bar{a})^n.$$

It is established experimentally that values of the coefficient of viscosity ω should be chosen from the range 1.0 to 1.5.

The difference scheme, Eqs. (5)-(7), has a second order of accuracy on sufficiently smooth solutions of system (1). The proof of this statement is effected by using expansions of the functions in Taylor series [1].

3. Analysis of the Stability of the Difference Scheme. The practical purpose for analyzing the stability of the difference equations is to determine the permissible step in time Δt^n. In order to analyze the stability, the equations are linearized, after which Fourier's method is applied to the linearized equations.

As a result of linearization of the general system (1), when $\nu = 0$, a system with constant coefficients is obtained:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0, \tag{8}$$

where

$$U = \begin{pmatrix} u_0 \\ v_0 \\ p_0 \\ \rho_0 \end{pmatrix};\quad A = \begin{pmatrix} u & 0 & \rho^{-1} & 0 \\ 0 & u & 0 & 0 \\ \rho\bar{a}^2 & 0 & u & 0 \\ \rho & 0 & 0 & u \end{pmatrix};\quad B = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & v & \rho^{-1} & 0 \\ 0 & \rho\bar{a}^2 & v & 0 \\ 0 & \rho & 0 & v \end{pmatrix}.$$

A similarity transformation exists with the matrix X, symmetrizing the linearized system of gasdynamic equations (8), i.e., such that the matrices of the coefficients of the equivalent system

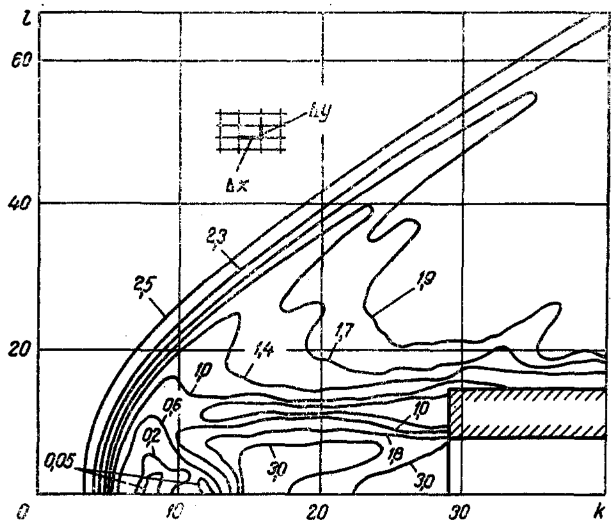

Fig. 3

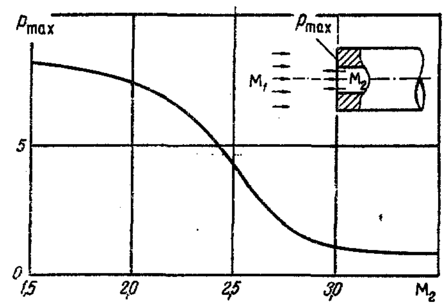

Fig. 4

Fig. 3. Field of Mach number isolines for the variant in Fig. 2.

Fig. 4. Dependence of the maximum pressure at the end surface on M_2, when $M_1 = 2.5$.

$$\frac{\partial V}{\partial t} + C\frac{\partial V}{\partial x} + D\frac{\partial V}{\partial y} = 0, \tag{9}$$

are symmetrical, where $V = XU$, $C = XAX^{-1}$, $D = X13X^{-1}$. The requirements of symmetrization determine unambiguously the transforming matrix X; its simplest form is the following

$$X = \begin{pmatrix} \rho\bar{a} & 0 & 0 & 0 \\ 0 & \rho\bar{a} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\bar{a}^2 \end{pmatrix}. \tag{10}$$

It is not difficult to prove that the norms of the matrix X and X^{-1} are bounded. In the future, therefore, it can be assumed that in system (8), A and B are symmetrical matrices.

The difference scheme (5)-(6), applied to the linear system (8), is written in the form of vector equations

$$U^{n+\frac{1}{2}}_{k+\frac{1}{2},\, l+\frac{1}{2}} = \left[E\mu_x\mu_y - \frac{\Delta t}{2\Delta x}A\mu_y\delta_x - \frac{\Delta t}{2\Delta y}B\mu_x\delta_y + \frac{\Delta t^2}{8\Delta x\Delta y}(AB + BA)\delta_x\delta_y\right]U^n_{k+\frac{1}{2},\, l+\frac{1}{2}}, \tag{11}$$

$$U^{n+1}_{k,l} = U^n_{k,l} - \left(\frac{\Delta t}{\Delta x}A\mu_y\delta_x + \frac{\Delta t}{\Delta y}B\mu_x\delta_y\right)U^{n+\frac{1}{2}}_{k,l}. \tag{12}$$

Equation (11) is the linearization of Eq. (5), and Eq. (12) can be considered as the linearized Eq. (6). E is a unit matrix.

The reaction of the linear difference equations (11) and (12) is investigated on simple harmonic oscillations, described by the function

$$U^n_{k,l} = \xi^n \exp[i(k\varphi + l\psi)], \tag{13}$$

where φ and ψ are the frequency parameters. Substitution of the vector function (13) in Eq. (11) and (12) leads to the relation

$$\xi^{n+1} = S\xi^n,$$

in which S — the matrix of transformation of the scheme — is defined by the expression

$$S = E - 2\left(a\sin\frac{\varphi}{2}\cos\frac{\psi}{2} + b\cos\frac{\varphi}{2}\sin\frac{\psi}{2}\right)^2 - 2i\left(a\sin\frac{\varphi}{2}\cos\frac{\psi}{2} + b\cos\frac{\varphi}{2}\sin\frac{\psi}{2}\right) \times \left(E\cos\frac{\varphi}{2}\cos\frac{\psi}{2} - \frac{ab+ba}{2}\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right), \tag{14}$$

where $a=\frac{\Delta t}{\Delta x}A;\ b=\frac{\Delta t}{\Delta y}B.$

For stability of the scheme, it is sufficient that the inequality

$$|(Sq,q)|^2\leqslant 1 \tag{15}$$

should be satisfied for all unit vectors q [1].

We introduce the notations:

$$S=R-iJ,\ R=E-K,\ K=2\Omega^2,\ J=2\Omega\Gamma,$$

$$\Omega=a\sin\frac{\varphi}{2}\cos\frac{\psi}{2}+b\cos\frac{\varphi}{2}\sin\frac{\psi}{2},$$

$$\Gamma=E\cos\frac{\varphi}{2}\cos\frac{\psi}{2}-\frac{ab+ba}{2}\sin\frac{\varphi}{2}\sin\frac{\psi}{2},$$

$$r=(Rq,q),\ j=(Jq,q),\ \alpha=|\Omega q|^2,\ \beta=|\Gamma q|^2.$$

Using the notations introduced, the left-hand side of Eq. (15) can be written in the form

$$|(Sq,q)|^2=r^2+j^2=(1-2\alpha)^2+4(\Omega\Gamma q,q)^2.$$

But in view of the symmetry of Ω and according to Schwartz's inequality

$$(\Omega\Gamma q,q)^2=(\Gamma q,\Omega q)^2\leqslant|\Gamma q|^2|\Omega q|^2=\alpha\beta.$$

Consequently,

$$|(Sq,q)|^2\leqslant(1-2\alpha)^2+4\alpha\beta=1-4\alpha(1-\alpha-\beta).$$

It remains to explain the conditions for satisfying the inequality $\alpha+\beta\leq 1$. As

$$(\alpha+\beta)^2=((\Gamma^2+\Omega^2),\ q,\ q)^2\leqslant|(\Gamma^2+\Omega^2)q|^2,$$

$$\Gamma^2+\Omega^2=\left(E\cos^2\frac{\varphi}{2}+a^2\sin^2\frac{\varphi}{2}\right)\cos^2\frac{\psi}{2}+\left(\left(\frac{ab+ba}{2}\right)^2\sin^2\frac{\varphi}{2}+b^2\cos^2\frac{\varphi}{2}\right)\sin^2\frac{\psi}{2},$$

the inequality

$$(\alpha+\beta)^2\leqslant\max(1,\ |aq|^2)\cos^2\frac{\psi}{2}+\max\left(\left|\frac{ab+ba}{2}q\right|^2,\ |bq|^2\right)\sin^2\frac{\psi}{2}\leqslant\max\left(1,\ |aq|^2,\ |bq|^2,\ \left|\frac{ab+ba}{2}q\right|^2\right).$$

is valid, from which follow the required conditions of stability

$$\|a\|\leqslant 1,\ \|b\|\leqslant 1. \tag{16}$$

In view of the symmetry of the matrices a and b, the conditions of Eq. (16) can be expressed in terms of eigenvalues and represented in a form which is suitable for use

$$\varkappa^n\leqslant 1,\ \lambda^n\leqslant 1. \tag{17}$$

From the fact that the schemes of [1], [3] and [4] are stable with the limitations $x^n \leq 1/\sqrt{2}$, $\lambda^n \leq 1/\sqrt{2}$, it follows that the scheme being described is more stable. Further relaxation of the inequality (17) is impossible, because the rate of transfer of perturbations in the difference solutions cannot be less than in the solution of the differential equations (Courant–Friedrichs–Levy condition). Consequently, the difference scheme has the maximum stability.

4. Results of the Calculations. The solution of the problem being considered, concerning the counterinteraction of supersonic flows has been carried out in dimensionless parameters, for the reduction to which the pressure p_1, density ρ_1, density ρ_1 and the radius of the cylinder r_1 are taken as independent units of scale. The 6 parameters M_1, M_2, p_2, ρ_2, r_0 and γ are chosen as controlling parameters.

In order to illustrate the possibilities of the method, Figs. 2 to 4 show a group of variants differing in values of M_2 (over the range from 1.5 to 3.5), with fixed values of the other parameters: $p_2 = 1$, $\rho_2 = 1$, $M_1 = 2.5$, $r_0 = 8/15$ and $\gamma = 1.4$. The variants shown are calculated on a network of 40×86 elements, so that the time of calculation of one variant, using this network, amounted to about 4 h (BÉSM-6).

Figure 2 shows the isobar field, formed as a result of the interaction of streams with $M_1 = 2.5$ and $M_2 = 3.25$. The magnitude of the egress of the head wave, the location and shape of the counter-wave and the dimensions of the reduced pressure zones can be estimated by the disposition of the isobars. The behavior of the isobars in the vicinity of the axis near the end of the cylinder affects the formation in the jet of an oblique shock wave.

Figure 3 represents the Mach number isolines for the same variant. The region of subsonic flow is bounded in this figure by two sonic lines ($M = 1$) each of which, having started at the surface of the cylinder, finishes at one of the shock waves — the leading or the counter shock wave.

The graph drawn in Fig. 4 shows how the maximum pressure at the end surface of the body depends on M_2. When $M_2 = 0$ in the given case, $p_{max} = 8.34$ and in the variant with $M_2 = 3.25$ it is 0.86. Thus, the counter flow cardinally changes the pattern of streamlined flow and can reduce significantly the pressure at the surface of the streamlined body.

NOTATION

y, radial coordinate; x, axial coordinate; t, time; p, pressure; ρ, density; u, axial velocity component; v, radial velocity component; e, total specific energy, $\rho(u^2 + v^2)/2 + p(\gamma - 1)$; γ, ratio of specific heats; $\bar{a}$, velocity of sound; M, Mach number; Δx, pitch of difference grid with respect to x; Δy, pitch of difference grid with respect to y; Δt^n, n-th time strip; k, axial index of node of difference grid; l, radial index of node of difference grid; p_1, ρ_1, M_1, parameters of the incident flow; p_2, ρ_2, M_2, parameters of the counter flow; r_1, radius of the body; r_2, channel radius; $r_0 = r_2/r_1$, ratio of radii.

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